

Tomography in abstract Hilbert spaces

V.I. Man'ko

P.N. Lebedev Physical Institute, Leninskii Prospect 53, Moscow 119991, Russia

G. Marmo, A. Simoni, F. Ventriglia

Dipartimento di Scienze Fisiche dell' Università "Federico II" e INFN, Sezione di Napoli

Complesso Universitario di Monte S. Angelo, via Cintia, 80126 Napoli, Italy

(e-mail: manko@na.infn.it, marmo@na.infn.it, simoni@na.infn.it, ventriglia@na.infn.it)

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Abstract

The tomographic description of a quantum state is formulated in an abstract infinite dimensional Hilbert space framework, the space of the Hilbert-Schmidt linear operators, with trace formula as scalar product. Resolutions of the unity, written in terms of over-complete sets of rank-one projectors and of associated Gram-Schmidt operators taking into account their non-orthogonality, are then used to reconstruct a quantum state from its tomograms. Examples of well known tomographic descriptions illustrate the exposed theory.

1 Introduction

Standard description of a quantum state is done by means of a vector in an abstract Hilbert space [1]. There are also descriptions on phase space of quasi-distributions like Wigner function [2], Husimi-Kano Q-function [3, 4], Sudarshan-Glauber P-function [5, 6], all of them are used to represent quantum states, both pure and mixed ones. Recently the optical probability distribution of homodyne quadrature has been introduced [7, 8] as an approach to reconstruct the Wigner function of a quantum state.

This approach has been extended and symplectic tomography of a quantum state has been suggested [9] for reconstructing the Wigner function. The description of a quantum states by means of a tomographic probability distribution (tomogram) has been used to suggest a new formulation of quantum mechanics [10] in which these probability distributions identify quantum states alternatively to vectors in a Hilbert space, or density operators for mixed states. Due to the development of tomographic methods, it turns out that there exists a map of the elements of a Hilbert space (vectors) onto elements of the set of the probability distributions (tomograms of a quantum state). The general problem

how to construct the tomographic map and what is the explicit mathematical mechanism providing this map in abstract Hilbert spaces has been studied in [11] for the finite dimensional case. The main idea of this mechanism is to consider the squared modulus of the scalar product of two vectors in the initial abstract Hilbert space \mathcal{H} as the standard scalar product of other vectors in another Hilbert space \mathbb{H} , which is the space of operators acting on the initial space \mathcal{H} . As it has been established in Ref. [11], the approach which uses two Hilbert spaces and the completeness condition, related to any (over-) complete set of rank-one projectors in the new Hilbert space \mathbb{H} , naturally provides the necessary ingredients to construct the tomographic map under discussion.

The aim of the present work is to review the mathematical mechanism of the tomographic map in finite-dimensional Hilbert spaces and to extend the construction to the case of infinite dimensional Hilbert spaces. We will clarify how the known examples of tomographic maps in the infinite dimensional Hilbert spaces, like symplectic tomography [12] and photon number tomography [13, 14, 15], correspond to our formulation for constructing a tomographic map in an abstract Hilbert space. Moreover we reconsider briefly the known non-negative quasi-distribution which is the Husimi-Kano Q -function from the point of view of “coherent state tomography”. As it is shown in Ref. [16], the construction of the Husimi-Kano Q -function can be interpreted as finding a specific tomographic set of basis vectors which provides the possibility to reconstruct the density operator in terms of the Q -function exactly in the framework of our tomographic approach in an abstract Hilbert space.

The paper is organized as follows. In the next section 2 the tomographic map in finite-dimensional abstract Hilbert space is reviewed. In section 3 the tomographic sets in Hilbert spaces are discussed in a general setting. In section 4 minimal tomographic sets and resolutions of identity are considered in infinite dimensional Hilbert spaces. Problem of operators generating tomographic sets and examples of squeeze tomography and symplectic tomography as well as a finite dimensional counter-example are considered in section 5. *Skewness* (i.e., completeness) of basis vectors, coherent state and photon number tomographies are discussed in section 6. Some conclusions and perspectives are drawn in section 7.

2 The finite dimensional case

In a previous work [11], we have provided an interpretation of quantum tomography in an abstract, finite-dimensional, Hilbert space \mathcal{H} in terms of complete sets of rank-one projectors $\{P_\mu\}_{\mu \in M}$, where M is a set of (multi-) parameters, discrete or continuous, collectively denoted by μ . In general, a tomogram of a quantum state $|\psi\rangle$ is a positive real number $\mathcal{T}_\psi(\mu)$ depending on the parameter μ which labels a set of states $|\mu\rangle \in \mathcal{H}$, defined as

$$\mathcal{T}_\psi(\mu) := |\langle \mu | \psi \rangle|^2. \quad (1)$$

Our main idea was to regard the tomogram $\mathcal{T}_\psi(\mu)$ as a scalar product on the (Hilbert) space \mathbb{H} of the rank-one projectors $|\mu\rangle\langle\mu| = P_\mu \rightarrow |P_\mu\rangle \in \mathbb{H}$:

$$\mathcal{T}_\psi(\mu) = \text{Tr}(P_\mu \rho_\psi) =: \langle P_\mu | \rho_\psi \rangle. \quad (2)$$

Equation (2) may readily be used to define the tomogram of any density operator $\hat{\rho}$ or any other (bounded) operator \hat{A}

$$\mathcal{T}_A(\mu) := \text{Tr}(P_\mu \hat{A}) = \langle P_\mu | \hat{A} \rangle. \quad (3)$$

Equation (3) shows in general that the tomogram of the operator \hat{A} may be thought of as a *symbol* of \hat{A} : in other words, by means of the set $\{|\mu\rangle\langle\mu|\}_{\mu \in M}$, to any operator \hat{A} a function $\langle \mu | \hat{A} | \mu \rangle$ of the variables collectively denoted by μ corresponds in a given functional space. For instance, in the case of the symplectic tomography, the variables μ vary in the phase space M of the physical system. So, a tomography may be thought of as a de-quantization, and in fact we found useful to study the quantum-classical transition by comparing classical limits of quantum tomograms with the corresponding classical tomograms [18]. Of course, while the correspondence $\hat{A} \rightarrow \mathcal{T}_A(\mu)$ may be thought of as a de-quantization, the inverse correspondence $\mathcal{T}_A(\mu) \rightarrow \hat{A}$ may be considered to give a quantization. The symbol determines completely the operator: the reconstruction of the operator \hat{A} from its tomogram $\mathcal{T}_A(\mu)$ may be written as:

$$\hat{A} = \sum_{\mu \in M} \hat{K}_\mu \text{Tr}(P_\mu \hat{A}) \Leftrightarrow |A\rangle = \sum_{\mu \in M} |K_\mu\rangle \langle P_\mu | A \rangle. \quad (4)$$

In other words, the reconstruction of any operator is possible because the tomographic set $\{P_\mu\}_{\mu \in M}$ provides a resolution of the identity (super-) operator¹ on \mathbb{H}

$$\hat{\mathbb{I}} = \sum_{\mu \in M} \hat{K}_\mu \text{Tr}(P_\mu \cdot) = \sum_{\mu \in M} |K_\mu\rangle \langle P_\mu|. \quad (5)$$

Here the \hat{K}_μ 's are Gram-Schmidt operators, which take into account that in general the projectors P_μ 's are not orthogonal, while the sum may be an integral with a suitable measure. Thus, for the finite n -dimensional case, $\mathbb{H} = \mathcal{H} \otimes \mathcal{H}$ is n^2 -dimensional and a *minimal* tomographic set is a basis $\{P_k\}$, $k \in \{1, \dots, n^2\}$, of rank-one projectors which may be orthonormalized by a Gram-Schmidt procedure

$$|V_j\rangle = \sum_{k=1}^{n^2} \gamma_{jk} |P_k\rangle \quad , \quad \langle V_i | V_j \rangle = \delta_{ij}. \quad (6)$$

In general, every element of the orthonormal basis $\{|V_j\rangle\}$ is a linear combination of projectors, rather than a single projector like $|P_k\rangle$. Then a resolution of the

¹In the present paper we do not address the problem of the continuity of the reconstruction formula [17], which is granted in all our examples.

unity on \mathbb{H} in terms of the P_k 's reads as

$$\hat{\mathbb{I}}_{n^2} = \sum_{i=1}^{n^2} |V_i\rangle \langle V_i| = \sum_{i,j,l=1}^{n^2} \gamma_{il}^* \gamma_{ij} P_j \text{Tr}(\hat{P}_l) = \sum_{l=1}^{n^2} |K_l\rangle \langle P_l| = \sum_{j=1}^{n^2} |P_j\rangle \langle K_j| \quad (7)$$

where the Gram-Schmidt operator \hat{K}_l has been introduced

$$|K_l\rangle = \sum_{i=1}^{n^2} \gamma_{il}^* |V_i\rangle = \sum_{i,j=1}^{n^2} \gamma_{il}^* \gamma_{ij} |P_j\rangle. \quad (8)$$

We observe that \hat{K}_l is a nonlinear function of the projectors P_k , because also the coefficients γ 's depend on the projectors. Moreover, it results

$$\langle P_i | K_l \rangle = \sum_{j=1}^{n^2} \gamma_{jl}^* \langle P_i | V_j \rangle = \sum_{j,k=1}^{n^2} \gamma_{jl}^* (\gamma^*)_{ik}^{-1} \langle V_k | V_j \rangle = \sum_{j=1}^{n^2} \gamma_{jl}^* (\gamma^*)_{ij}^{-1} = \delta_{il}. \quad (9)$$

Similar formulae hold even for any other tomographic, i.e. (over-) complete, set $\{P_\mu\}_{\mu \in M}$. For instance for the spin tomography, in the *maximal* qu-bit case $M = S^2$ is the Bloch sphere of all rank-one projectors and we have [11]:

$$\hat{\mathbb{I}} = \int_0^{2\pi} \int_0^\pi |K(\theta, \phi)\rangle \text{Tr}(P(\theta, \phi) \cdot) \sin \theta d\theta d\phi, \quad (10)$$

where, in matrix form,

$$P(\theta, \phi) = \frac{1}{2} [\mathbb{I} + \vec{n} \cdot \vec{\sigma}] = \frac{1}{2} \begin{bmatrix} 1 + \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & 1 - \cos \theta \end{bmatrix} \quad (11)$$

and

$$\hat{K}(\theta, \phi) = \frac{1}{4\pi} \begin{bmatrix} 1 + 3 \cos \theta & 3e^{-i\phi} \sin \theta \\ 3e^{i\phi} \sin \theta & 1 - 3 \cos \theta \end{bmatrix}, \quad (12)$$

so that, for any operator \hat{A} , it results

$$\begin{aligned} \hat{A} &= \int_0^{2\pi} \int_0^\pi \hat{K}(\theta, \phi) \text{Tr}(P(\theta, \phi) \hat{A}) \sin \theta d\theta d\phi \\ &= \int_0^{2\pi} \int_0^\pi P(\theta, \phi) \text{Tr}(\hat{K}(\theta, \phi) \hat{A}) \sin \theta d\theta d\phi. \end{aligned} \quad (13)$$

This example shows that the orthogonality relations of the minimal case, Eq. (9), do not hold in general.

After this brief introductory sketch of our previous work, we are ready to extend our interpretation of tomography in abstract, infinite dimensional Hilbert spaces

3 Tomographic sets in Hilbert spaces

Let M be a set of (multi-) parameters μ , and assign a map

$$\mu \in M \longrightarrow P_\mu \in \mathbb{P} \subset \mathbb{H} \quad (14)$$

from M into the set \mathbb{P} of all the rank-one projectors of the Hilbert space \mathbb{H} . By definition, the set $\{P_\mu\}_{\mu \in M}$ is *tomographic* if it is complete in \mathbb{H} . A tomographic set determines a *tomography* which is a functional, linear in the second argument

$$\mathcal{T} : \mathbb{P} \times \mathbb{H} \longrightarrow \mathbb{C}, (P_\mu, A) \longrightarrow \mathcal{T}_A(\mu) = \text{Tr} \left(P_\mu \hat{A} \right) = \langle P_\mu | A \rangle. \quad (15)$$

This definition is appropriate in the finite n -dimensional case, where

$$|\mu\rangle \in \mathcal{H}_n \Leftrightarrow P_\mu \in \mathbb{H}_{n^2} = B(\mathcal{H}_n) = \mathcal{H}_n \otimes \mathcal{H}_n, \quad (16)$$

but in the infinite dimensional case more care is needed, because the relation $\mathbb{H} = B(\mathcal{H})$ is no more valid. On the contrary, there are several relevant spaces [19, 20], as the space of bounded operators $B(\mathcal{H})$ and that of compact operators $C(\mathcal{H})$, the space of Hilbert-Schmidt operators \mathfrak{J}_2 and that of trace-class operators \mathfrak{J}_1 . Their mutual relations are:

$$\mathfrak{J}_1 \subset \mathfrak{J}_2 \subset C(\mathcal{H}) \subset B(\mathcal{H}). \quad (17)$$

Besides, we recall that $B(\mathcal{H})$ is a Banach space and $C(\mathcal{H})$ a Banach subspace, with the uniform norm $\|A\| = \sup_{(\|\psi\|=1)} \|A\psi\|$, while \mathfrak{J}_2 is a Hilbert space with scalar product $\langle A|B \rangle = \text{Tr} (A^\dagger B)$ and norm $\|A\|_2 = \sqrt{\text{Tr} (A^\dagger A)}$. Finally \mathfrak{J}_1 , which is not closed in $B(\mathcal{H})$ with the uniform norm, is a Banach space with the norm $\|A\|_1 = \text{Tr} (|A|)$. The following inequalities hold true

$$\|A\| \leq \|A\|_2 \leq \|A\|_1. \quad (18)$$

So \mathfrak{J}_2 , the only Hilbert space at our disposal to implement our definition of tomographic set, is endowed with a topology which, when restricted to the trace-class operators, is not equivalent to the topology of \mathfrak{J}_1 . This may have serious consequences. In fact, in the finite dimensional case, the set $\{P_\mu\}_{\mu \in M}$ is complete iff

$$\text{Tr} (P_\mu A) = 0 \quad \forall \mu \in M \implies A = 0. \quad (19)$$

Such a condition guarantees the full reconstruction of any observable from its tomograms.

Now, in \mathfrak{J}_2 , Eq. (19) reads

$$\langle P_\mu | A \rangle = 0 \quad \forall \mu \in M \implies A = 0 \quad \& \quad A \in \mathfrak{J}_2. \quad (20)$$

Then, as \mathfrak{J}_2 is a $*$ -ideal in $B(\mathcal{H})$, it may exists a non-zero operator B , which is bounded but not Hilbert-Schmidt, such that

$$\text{Tr} (P_\mu B) = 0 \quad \forall \mu \in M \quad (21)$$

In that case, a non-ambiguous reconstruction of two different observables is impossible when their difference is an operator like B . In other words, different observables may be tomographically separated only when their difference is Hilbert-Schmidt. For a deeper discussion, see Ref.[21].

Nevertheless there is a second case, when the set $\{P_\mu\}_{\mu \in M}$ of trace-class operators is complete even in \mathfrak{I}_1 . Then, recalling [22, 23] that \mathfrak{I}_1 is a $*$ -ideal in its dual space $B(\mathcal{H})$:

$$\mathfrak{I}_1^* = B(\mathcal{H}), \quad (22)$$

the expression $\text{Tr}(P_\mu A)$ is nothing but the value of the linear functional $\text{Tr}(\cdot A)$ in P_μ . Hence, Eq.(19) holds unconditionally

$$\text{Tr}(P_\mu A) = 0 \quad \forall \mu \in M \implies 0 = \|\text{Tr}(\cdot A)\| = \|A\| \implies A = 0. \quad (23)$$

Clearly, this second case is more general: the tomographic map is finer and is able to better distinguish different observables.

Thus, the finest tomographies are those based on sets of rank-one projectors which are complete both in \mathfrak{I}_2 and in \mathfrak{I}_1 . As a matter of fact, this is the case for the main tomographic sets, as the photon number and the symplectic tomographic sets.

After the discussion of some of the topological subtleties of the infinite dimensional case, we are now ready to study an example, which allows for the construction of a *minimal* tomographic set, i.e., a basis of rank-one projectors.

4 Example

4.1 A (minimal) tomographic set spanning both \mathfrak{I}_2 and \mathfrak{I}_1

Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of an Hilbert space \mathcal{H} . Now we switch to the Dirac notation, $e_n \longleftrightarrow |n\rangle$, and get an orthonormal basis $\{E_{nm}\} = \{|n\rangle\langle m|\}_{n,m=1}^\infty$ of \mathfrak{I}_2 . In the basis $\{|n\rangle\}$, we have

$$(E_{nm})_{jk} = \langle j | E_{nm} | k \rangle = \delta_{jn} \delta_{mk} \quad ; \quad (24)$$

$$\text{Tr}(E_{qp}^\dagger E_{nm}) = \sum_{jk} \langle j | E_{qp}^\dagger | k \rangle \langle k | E_{nm} | j \rangle = \sum_{jk} \delta_{jp} \delta_{qk} \delta_{kn} \delta_{mj} = \delta_{qn} \delta_{pm} \quad . \quad (25)$$

A Hermitian orthogonal basis may be constructed with the compact operators

$$E_{nm}^+ = \frac{1}{2} (E_{nm} + E_{nm}^\dagger) \quad (n \leq m); \quad E_{nm}^- = \frac{i}{2} (E_{nm} - E_{nm}^\dagger) \quad (n > m). \quad (26)$$

The Hermitian basis is readily diagonalizable: for $n \neq m$ the set $\{E_{nm}^+, E_{nm}^-\}_{n,m}$ is isospectral, with simple eigenvalues $\pm 1/2$ and respective eigenvectors

$$|\Psi_{nm}^{+,\pm}\rangle = \frac{1}{\sqrt{2}} (|n\rangle \pm |m\rangle) \quad ; \quad |\Psi_{nm}^{-,\pm}\rangle = \frac{1}{\sqrt{2}} (|m\rangle \pm i |n\rangle), \quad (27)$$

where \pm label the eigenvalues. Their associated projectors

$$P_{nm}^{+,\pm} = |\Psi_{nm}^{+,\pm}\rangle \langle \Psi_{nm}^{+,\pm}|, \quad P_{nm}^{-,\pm} = |\Psi_{nm}^{-,\pm}\rangle \langle \Psi_{nm}^{-,\pm}|, \quad (28)$$

together with the diagonal ($n = m$) projectors

$$P_{nn} = |n\rangle \langle n|, \quad (29)$$

are a tomographic set. In fact, as

$$P_{nm}^{+,\pm} = \frac{1}{2}(P_{nn} + P_{mm}) \pm E_{nm}^+ \quad , \quad P_{nm}^{-,\pm} = \frac{1}{2}(P_{nn} + P_{mm}) \pm E_{nm}^-, \quad (30)$$

the set contains a basis of \mathfrak{I}_2 of rank-one projectors. Moreover, the set is complete in \mathfrak{I}_1 . In fact, assume that the linear functional $\text{Tr}(A \cdot)$, with $A \in B(\mathcal{H})$, vanishes on the tomographic set. Then

$$\text{Tr}(AP_{nn}) = \langle n|A|n\rangle = 0 \quad \forall n, \quad (31)$$

so that the diagonal matrix elements of A are zero. Bearing this in mind, we have

$$\text{Tr}(AP_{nm}^{+,\pm}) = \frac{1}{2}\text{Tr}(A(P_{nn} + P_{mm} \pm 2E_{nm}^+)) = \pm \frac{1}{2}(\langle m|A|n\rangle + \langle n|A|m\rangle) = 0,$$

$$\text{Tr}(AP_{nm}^{-,\pm}) = \frac{1}{2}\text{Tr}(A(P_{nn} + P_{mm} \pm 2E_{nm}^-)) = \pm \frac{i}{2}(\langle m|A|n\rangle - \langle n|A|m\rangle) = 0,$$

which yield

$$\langle m|A|n\rangle = 0 \quad \forall m, n \Leftrightarrow A = 0, \quad (32)$$

so that A is the zero operator.

Finally we observe that a *minimal* tomographic set, i.e. a basis of rank-one projectors, may be chosen by taking just one projector from each pair $P_{nm}^{+,\pm}$ with $n < m$, only one projector from each pair $P_{nm}^{-,\pm}$ with $n > m$ and all the diagonal P_{nn} 's. Such a minimal set is obviously complete both in \mathfrak{I}_2 and in \mathfrak{I}_1 .

4.2 The corresponding resolution of the unity

We now evaluate explicitly the resolution of unity determined by the full (non-minimal) set of projectors. To do this, we start from the representation of a (bounded) operator B as

$$B = \sum_{n,m} \langle n|B|m\rangle |n\rangle \langle m|. \quad (33)$$

In view of the decomposition of any operator as a sum of two selfadjoint operators

$$B = \frac{1}{2}(B + B^\dagger) - i\left(\frac{i}{2}(B - B^\dagger)\right), \quad (34)$$

we may assume B selfadjoint. Then the identity holds:

$$\begin{aligned}\langle n|B|m\rangle &= \frac{1}{2} [\langle \Psi_{nm}^{+,+} | B | \Psi_{nm}^{+,+} \rangle - \langle \Psi_{nm}^{+,-} | B | \Psi_{nm}^{+,-} \rangle] \\ &+ \frac{i}{2} [\langle \Psi_{nm}^{-,+} | B | \Psi_{nm}^{-,+} \rangle - \langle \Psi_{nm}^{-,-} | B | \Psi_{nm}^{-,-} \rangle].\end{aligned}\quad (35)$$

In other terms:

$$\langle n|B|m\rangle = \frac{1}{2} [\text{Tr}(BP_{nm}^{+,+}) - \text{Tr}(BP_{nm}^{+,-}) + i(\text{Tr}(BP_{nm}^{-,+}) - \text{Tr}(BP_{nm}^{-,-}))].$$

Thus, we get the reconstruction formula

$$B = \sum_{n,m} \frac{1}{2} |n\rangle \langle m| [\text{Tr}(BP_{nm}^{+,+}) - \text{Tr}(BP_{nm}^{+,-}) + i\text{Tr}(BP_{nm}^{-,+}) - i\text{Tr}(BP_{nm}^{-,-})],$$

or, equivalently,

$$\begin{aligned}B &= \sum_n P_{nn} \text{Tr}(BP_{nn}) + \sum_{n<m} E_{nm}^+ [\text{Tr}(BP_{nm}^{+,+}) - \text{Tr}(BP_{nm}^{+,-})] \\ &+ \sum_{n<m} E_{nm}^- [\text{Tr}(BP_{nm}^{-,+}) - \text{Tr}(BP_{nm}^{-,-})].\end{aligned}\quad (36)$$

Upon introducing a third label α to enumerate the $P^{\pm,\pm}$'s, we obtain the resolution of the unity as

$$\hat{\mathbb{I}} = \sum_n |P_{nn}\rangle \langle P_{nn}| + \sum_{n<m,\alpha} |K_{nm}^\alpha\rangle \langle P_{nm}^\alpha|, \quad (37)$$

where

$$|K_{nm}^{+,\pm}\rangle = \pm E_{nm}^+, \quad |K_{nm}^{-,\pm}\rangle = \pm E_{nm}^-. \quad (38)$$

5 Families of operators generating tomographic sets

An interesting question is how to construct tomographic sets. We will answer this question by considering how some of the main tomographic sets are generated, so providing also a few well known examples to discuss.

We start with a fiducial Hermitian operator \hat{T}_0 and act on it with a family of unitary operators $\{U_\mu\}$, depending on some parameters $\mu \in M$, to generate a family of (iso-spectral) Hermitian operators

$$\hat{T}_\mu = U_\mu \hat{T}_0 U_\mu^\dagger. \quad (39)$$

Assuming \hat{T}_0 to be generic, i.e. with simple eigenvalues, the action of U_μ on the rank-one projectors associated with the eigenstates $\{|\psi_n^0\rangle\}_{n \in \mathbb{N}}$ of \hat{T}_0 gives rise to

a set of projectors, corresponding to the eigenstates $\{|\psi_{\mu,n}\rangle\}_n = \{U_\mu |\psi_n^0\rangle\}_n$ of \hat{T}_μ

$$P_{\mu,n} = U_\mu P_n^0 U_\mu^\dagger, \quad \mu \in M. \quad (40)$$

We observe that Eq. (40) suggests one could start with a fiducial rank-one projector P_0 as operator \hat{T}_0 and act on it with the unitary family to generate a set of projectors. However the use of a generic operator \hat{T}_0 allows, if the set $\{P_{\mu,n}\}, (\mu,n) \in M \times \mathbb{N}$, is tomographic, to obtain at once that the tomograms of any density operator $\hat{\rho}$ satisfy

$$\sum_n \mathcal{T}_\rho(\mu,n) = \sum_n \text{Tr}(\hat{\rho} P_{\mu,n}) = \sum_n \langle \psi_{\mu,n} | \hat{\rho} | \psi_{\mu,n} \rangle = 1, \quad \forall \mu \in M. \quad (41)$$

Such an identity is of capital importance, because it allows for the probabilistic interpretation of the tomographic map \mathcal{T} , as for any given μ the tomogram $\mathcal{T}_\rho(\mu)$ is a marginal probability distribution.

Then the question is, how the operator \hat{T}_0 and the unitary family $\{U_\mu\}$ have to be chosen to generate a tomographic set $\{P_{\mu,n}\}$? Or more simply, when is the set $\{P_{\mu,n}\}$ tomographic?

We may preliminarily state a negative answer.

Proposition. *The set $\{P_{\mu,n}\}$ is not tomographic, if it exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ invariant under the action of both the operator \hat{T}_0 and the unitary family $\{U_\mu\}$.*

Proof. In fact, the set $\{P_{\mu,n}\}$ is not complete, as the non-zero operator $|\varphi_1\rangle\langle\varphi_2|$, with $|\varphi_1\rangle \in \mathcal{H}_1$ and $|\varphi_2\rangle \in \mathcal{H}_2$, is orthogonal to the whole set $\{P_{\mu,n}\}$:

$$\text{Tr}(|\varphi_1\rangle\langle\varphi_2| P_{\mu,n}) = \text{Tr}(|\varphi_1\rangle\langle\varphi_2| U_\mu |\psi_n^0\rangle\langle\psi_n^0| U_\mu^\dagger) = 0 \quad \forall \mu, n \quad (42)$$

because $\langle\varphi_2| U_\mu |\psi_n^0\rangle = 0$ or $\langle\psi_n^0| U_\mu^\dagger |\varphi_1\rangle = 0$, according to the case $|\psi_n^0\rangle \in \mathcal{H}_1$ or $|\psi_n^0\rangle \in \mathcal{H}_2$. ■

Example: the squeeze “tomography”. It is generated by the family of (iso-spectral) Hermitian operators depending on two real parameters

$$\hat{T}_{sq}(\mu, \nu) = S(\mu, \nu) \hat{a}^\dagger \hat{a} S^\dagger(\mu, \nu), \quad \mu, \nu \in \mathbb{R}, \quad (43)$$

where the unitary operators $\{S(\mu, \nu)\}$ depend quadratically on the harmonic oscillator creation and annihilation operators \hat{a}^\dagger, \hat{a} , see *Ref(Squeeze)*. Then, both the fiducial operator $\hat{a}^\dagger \hat{a}$ and the unitary family $\{S(\mu, \nu)\}$ commute with the Parity operator. So, the squeeze “tomography” is not a true tomography. Nevertheless, we can get a true tomographic set by a restriction to the subspace of even wave functions. Only there the existence of an inversion formula is granted. □

This example shows that the answer depends on a joint property of \hat{T}_0 and $\{U_\mu\}$, i.e. the relation between the commutants \hat{T}_0' of the fiducial operator and $\{U_\mu\}'$ of the unitary family.

So, we may restate the previous proposition as a *necessary condition*:

Proposition. *If the set $\{P_{\mu,n}\}$ is tomographic, then the family $\{\hat{T}_0, \{U_\mu\}\}$ is irreducible or, equivalently, the intersection of the commutants \hat{T}'_0 and $\{U_\mu\}'$ is trivial: $\hat{T}'_0 \cap \{U_\mu\}' = \{1\}$.*

For instance, by changing the unitary family $\{S(\mu, \nu)\}$ or the starting operator $\hat{a}^\dagger \hat{a}$ we may obtain from Eq.(43) tomographic families of selfadjoint operators, as in the following

Example: the symplectic tomography. It is generated by the same family of unitary operators $\{S(\mu, \nu)\}$ of the squeeze “tomography”, with the position operator \hat{Q} as fiducial operator:

$$\hat{T}(\mu, \nu) = S(\mu, \nu) \hat{Q} S^\dagger(\mu, \nu) = \mu \hat{Q} + \nu \hat{P}, \quad \mu, \nu \in \mathbb{R} \quad (44)$$

where \hat{P} is the momentum operator. The spectrum is continuous. The (improper) eigenvectors $\{|X\mu\nu\rangle\}$ of $\hat{T}(\mu, \nu)$ stem out from those of the position: $\hat{Q}|X\rangle = X|X\rangle, X \in \mathbb{R}$. In the position representation $\{|q\rangle\}$, for $\nu \neq 0$:

$$\langle q|X\mu\nu\rangle = \langle q|S(\mu, \nu)|X\rangle = \frac{1}{\sqrt{2\pi|\nu|}} \exp\left[-i\left(\frac{\mu}{2\nu}q^2 - \frac{X}{\nu}q\right)\right], \quad (45)$$

with

$$\langle X'\mu\nu|X\mu\nu\rangle = \delta(X - X') \quad (46)$$

Then the resolution of unity in matrix form reads [11]:

$$\int \frac{dX}{2\pi} d\mu d\nu \langle y|\exp\left[i\left(X - \mu\hat{Q} - \nu\hat{P}\right)\right]|y'\rangle \langle q'|X\mu\nu\rangle \langle X\mu\nu|q\rangle = \delta(q-y)\delta(q'-y'). \square$$

The irreducibility condition of $\{\hat{T}_0, \{U_\mu\}\}$ is too poor to get a sufficient condition. For this, more hypotheses must be added. For instance, the family of unitary operators $\{U_\mu\}$ may be chosen as a representation of a group $G : \mu \leftrightarrow g \in G$. When this representation is analytic in some neighborhood of $\mu = 0$, then the set $\{P_{\mu,n}\}$ is tomographic. This is the case of the coherent state and of the photon number tomographic sets, discussed in the next section. The analyticity condition may be substituted by other weaker hypotheses, but these further conditions are needed, as the following finite dimensional counter-example shows.

Example. On $\mathcal{H} = \mathbb{C}^2$, take for $\{U_\mu\}$ the family

$$U_1 = \hat{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_{-1} = \hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (47)$$

which represents the group Z_2 . As fiducial operator choose

$$\hat{T}_0 = \begin{bmatrix} \alpha & \beta \\ \beta^* & \gamma \end{bmatrix}, \quad \alpha, \gamma \in \mathbb{R}, \quad \beta \neq 0, \quad (48)$$

whose eigenvalues are $\lambda = [\alpha + \gamma \pm \sqrt{(\alpha - \gamma)^2 + 4|\beta|^2}]/2$. The condition $\beta \neq 0$ implies that $\hat{T}'_0 \cap \{U_\mu\}' = \{1\}$, so that $\{\hat{T}_0, \{U_\mu\}\}$ is irreducible. The isospectral

family is

$$\hat{T}_0 = \begin{bmatrix} \alpha & \beta \\ \beta^* & \gamma \end{bmatrix}, \quad \hat{P}\hat{T}_0\hat{P} = \begin{bmatrix} \alpha & -\beta \\ -\beta^* & \gamma \end{bmatrix}. \quad (49)$$

Then, if β is real and $\alpha = \gamma = 0$, $\hat{P}\hat{T}_0\hat{P} = -\hat{T}_0$ and the family $\{U_\mu\}$ does not displace \hat{T}_0 . Otherwise, the isospectral family has two different operators. But from Ref. [11] we know that three different operators are needed to get a basis of rank-one projectors. In any case, the set of the projectors associated with the eigenvectors of the isospectral family is *not* tomographic. \square

This simple example shows that, starting from \hat{T}_0 , the unitary family has to generate a number of different isospectral operators \hat{T}_μ sufficient to get a complete set of rank-one (eigen-) projectors $\{P_\mu\}$. So, the strong condition of analyticity is only a suitable way to obtain such a complete set. However, the unitary family needs not to be a representation of any group, as the case of the countable tomographic set of section 4 shows:

Example. Take as fiducial operator

$$\hat{T}_0 = \text{diag}[1, -1, 0, \dots, 0, \dots]. \quad (50)$$

The first 2×2 block is σ_3 , one of the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (51)$$

By means of the rules

$$[\sigma_j, \sigma_k]_+ = 2\delta_{jk}; \quad [\sigma_j, \sigma_k] = 2i\varepsilon_{jkl}\sigma_l \quad (52)$$

we get

$$\exp\left[-i\vec{\sigma} \cdot \vec{n} \frac{\phi}{2}\right] = \cos \frac{\phi}{2} - i\vec{\sigma} \cdot \vec{n} \sin \frac{\phi}{2} \quad (53)$$

so that

$$\exp\left[-i\vec{\sigma} \cdot \vec{n}_1 \frac{\pi}{2}\right] \sigma_3 \exp\left[i\vec{\sigma} \cdot \vec{n}_1 \frac{\pi}{2}\right] = \sigma_1, \quad \vec{n}_1 = \frac{1}{\sqrt{2}}(1, 0, 1), \quad (54)$$

and

$$\exp\left[-i\vec{\sigma} \cdot \vec{n}_2 \frac{\pi}{4}\right] \sigma_1 \exp\left[i\vec{\sigma} \cdot \vec{n}_2 \frac{\pi}{4}\right] = \sigma_2, \quad \vec{n}_2 = (0, 0, 1) \quad (55)$$

We may define the unitary operators $S(\vec{n}, \frac{\phi}{2})$ by embedding $\exp\left[-i\vec{\sigma} \cdot \vec{n} \frac{\phi}{2}\right]$ as first 2×2 block into a zero matrix.

Then, consider the unitary selfadjoint commuting operators $U_{1,n}$ and $U_{2,m}$ which interchange the vector components $1, n$ and $2, m$ respectively. Upon multiplying S 's and U 's operators, we construct a family (not a group) of unitary operators which displace the operator \hat{T}_0 of Eq.(50) and generate the tomographic family of compact operators $\{E_{n,m}^+, E_{n,m}^-\}$ of section 4:

$$\begin{aligned} U_{1,n} U_{2,m} S(\vec{n}_1, \frac{\pi}{2}) \hat{T}_0 S^\dagger(\vec{n}_1, \frac{\pi}{2}) U_{2,m} U_{1,n} &= 2E_{n,m}^+, \\ U_{1,n} U_{2,m} S(\vec{n}_2, \frac{\pi}{4}) S(\vec{n}_1, \frac{\pi}{2}) \hat{T}_0 S^\dagger(\vec{n}_1, \frac{\pi}{2}) S^\dagger(\vec{n}_2, \frac{\pi}{4}) U_{2,m} U_{1,n} &= 2E_{n,m}^-. \quad \square \end{aligned} \quad (56)$$

These last two examples show that neither the hypothesis of irreducibility nor the condition of analyticity of the representation $\{U_\mu\}$ of the group are necessary. However, the analytic dependence on the parameter μ together with the irreducibility of $\{\hat{T}_0, \{U_\mu\}\}$ are sufficient for constructing a tomographic set, as it is elucidated in the next section.

6 Skewness

From a geometrical point of view, tomographic sets are sets of “skew” projectors. In other words, the *skewness* of the projectors’ set denotes its completeness from a geometrical point of view. For instance, in the qu-bit case of the spin tomography, the manifold of rank-one projectors in the real space \mathbb{R}^4 of Hermitian operators is the Bloch sphere $S^2 : \left\{ (x^2)^2 + (x^3)^2 + (x^4)^2 = 1/4 \right\}$ placed in the plane $x^1 = 1/2$. Then any set of four points of S^2 , not lying on the equator, is skew as the corresponding projectors generate the whole space [11]. Then we give the following:

Definition. A set of projectors is globally skew when it spans the whole Hilbert space.

Thus, any tomographic set is *globally skew* as it is complete. Besides:

Definition. A set γ of projectors containing P_0 is locally skew in P_0 if any neighborhood of P_0 contains a skew subset of γ .

Back to the qu-bit case, any set of points on S^2 , not lying on the equator and with a limit point P_0 , is locally skew in P_0 . For the infinite dimensional case, we observe that the countable tomographic set of section 4 is skew globally but not locally. Perhaps the simplest case of a tomographic set which is skew globally and locally is provided by the following

Example: the coherent state tomography. This tomographic set, which is studied in Ref. [16], is generated by the displacement operators $\{\mathcal{D}(\alpha)\}$ depending on a complex parameter α

$$\mathcal{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}), \quad \alpha \in \mathbb{C}, \quad (57)$$

which acting on the projector $|0\rangle\langle 0|$ of the vacuum Fock state, $\hat{a}|0\rangle = 0$, yield the projectors

$$|\alpha\rangle\langle\alpha| = \mathcal{D}(\alpha)|0\rangle\langle 0|\mathcal{D}(\alpha)^\dagger, \quad \alpha \in \mathbb{C}, \quad (58)$$

associated to the usual coherent states

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \exp(\alpha \hat{a}^\dagger) \exp(-\alpha^* \hat{a}) |0\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} \hat{a}^{\dagger j} |0\rangle. \quad (59)$$

We recall that the coherent states are a (over-) complete set in the Hilbert space \mathcal{H} . Any bounded set containing a limit point α_0 in the complex α -plane defines a complete set of coherent states containing a limit point, the coherent state $|\alpha_0\rangle$, in the Hilbert space \mathcal{H} . In particular, any Cauchy sequence $\{\alpha_k\}$ of

complex numbers defines a Cauchy sequence of coherent states $\{|\alpha_k\rangle\}$, which is a complete set. The same holds for any extracted subsequence. This completeness property holds as $\exp(|\alpha|^2/2) \langle \alpha | \psi \rangle$ is an entire analytic function of the complex variable α^* , for any $|\psi\rangle \in \mathcal{H}$, with a non-isolated zero in α_0^* . Then

$$\langle \alpha_k | \psi \rangle = 0 \quad \forall k \Rightarrow |\psi\rangle = 0. \quad (60)$$

Besides, any bounded operator A may be completely reconstructed from its diagonal matrix elements $\langle \alpha_k | A | \alpha_k \rangle$. In fact, $\exp(|\alpha|^2/2 + |\beta|^2/2) \langle \alpha | A | \beta \rangle$ is an analytical function of the complex variables α^*, β , so it is uniquely determined by its value $\exp(|\alpha|^2) \langle \alpha | A | \alpha \rangle$ on the diagonal $\beta = \alpha$. This is an entire function of the real variables $\Re\alpha, \Im\alpha$, which is in turn uniquely determined by its values on any set with an accumulation point.

The rank-one projectors associated to a complete set of coherent states are complete in the Hilbert space \mathfrak{H}_2 . In particular, any Cauchy sequence $\{|\alpha_k\rangle\}$ generates a tomographic set $\{|\alpha_k\rangle \langle \alpha_k|\}$. In fact, bearing in mind the previous remark on the reconstruction of a bounded operator, it results

$$\text{Tr}(A |\alpha_k\rangle \langle \alpha_k|) = \langle \alpha_k | A | \alpha_k \rangle = 0 \quad \forall k \Rightarrow A = 0 \quad \& \quad A \in B(\mathcal{H}). \quad (61)$$

This shows that a tomographic set of coherent state projectors is complete even in \mathfrak{H}_1 . So it is globally skew. Moreover, any extracted subsequence $\{|\alpha_{k_j}\rangle \langle \alpha_{k_j}|\}$ is again complete, so $\{|\alpha_k\rangle \langle \alpha_k|\}$ is locally skew in its limit point. The case when α varies in the whole complex plane and the associated reconstruction formula are discussed in [16]. \square

The same considerations hold for the following example, strictly connected to the previous one.

Example: the photon number tomography. It is generated by the irreducible family $\{\hat{a}^\dagger \hat{a}, \{\mathcal{D}(\alpha)\}\}$, where the displacement operators $\{\mathcal{D}(\alpha)\}$ act on the same fiducial operator $\hat{a}^\dagger \hat{a}$ of the squeeze “tomography”,

$$\hat{T}(\alpha) = \mathcal{D}(\alpha) \hat{a}^\dagger \hat{a} \mathcal{D}(\alpha)^\dagger, \quad \alpha \in \mathbb{C} \quad (62)$$

The family of selfadjoint operators $\hat{T}(\alpha)$ has the spectrum of the number operator $\hat{a}^\dagger \hat{a}$, and eigenvectors $|n\alpha\rangle = \mathcal{D}(\alpha) |n\rangle$. With $\alpha = (\nu + i\mu)/\sqrt{2}$, in the position representation $\langle y | n\alpha \rangle$ is

$$\begin{aligned} \int dq \langle y | \mathcal{D}(\alpha) | q \rangle \langle q | n \rangle &= \int dq \delta(y - q - \nu) \exp[i(\mu q + \mu\nu/2)] \langle q | n \rangle \\ &= \exp[i(\mu y - \mu\nu/2)] \langle y - \nu | n \rangle, \end{aligned} \quad (63)$$

where the n -th Hermite function $\langle q | n \rangle$ is

$$\langle q | n \rangle = (\sqrt{\pi} 2^n n!)^{-1/2} \exp(-\frac{1}{2}q^2) H_n(q). \quad (64)$$

We recall that the photon number projectors' set, containing the complete set of the coherent state projectors, is a tomographic set complete both in \mathfrak{H}_2

and \mathfrak{I}_1 . For the same reason, any Cauchy sequence $\{|n\alpha_k\rangle\langle n\alpha_k|\}$ is locally skew in its limit point.

The whole set of photon number projectors generates the resolution of unity

$$\mathbb{I} = \sum_{n=0}^{\infty} \int \frac{d^2\alpha}{\pi} K^{(s)}(n, \alpha) \text{Tr}(|n\alpha\rangle\langle n\alpha| \cdot) \quad (65)$$

The Gram-Schmidt operator $K^{(s)}$ is given by

$$K^{(s)}(n, \alpha) = \frac{4}{s^2 - 1} \left(\frac{s+1}{s-1} \right)^n \mathcal{D}(\alpha) \left(\frac{s-1}{s+1} \right)^{\hat{a}^\dagger \hat{a}} \mathcal{D}^\dagger(\alpha). \quad (66)$$

Here s is a real parameter, $-1 < s < 1$, which labels the family of equivalent kernels $K^{(s)}(n, \alpha)$. This formula corrects the corresponding expressions given in [11].

The check of the matrix form of the resolution of the unity, Eq.(65), in the position representation is done in [16] and yields:

$$\sum_{n=0}^{\infty} \int \frac{d^2\alpha}{\pi} \langle y' | n\alpha \rangle \langle n\alpha | x' \rangle \langle x | K^{(s)}(n, \alpha) | y \rangle = \delta(x - x') \delta(y - y'), \quad (67)$$

for any allowed s , as it was expected. \square

7 Conclusions

To conclude we summarize the main points of the paper. We have reviewed the tomographic methods to map the vectors (and non-negative Hermitian trace-class operators) in abstract Hilbert spaces onto standard probability distributions and established conditions for the existence of the inverse transform both for the finite and infinite-dimensional cases.

In the infinite-dimensional case all the known examples of tomographies, like symplectic tomography, coherent state tomography, photon number tomography, squeeze tomography, were considered in the suggested framework of the existence of tomographic sets as over-complete bases of rank-one projectors. Any such a basis determines a completeness relation, that is a resolution of the (super-) identity operator, acting on the space of the bounded operators on the initial Hilbert space \mathcal{H} , which is expressed generally in terms of the rank-one projectors and the corresponding Gram-Schmidt operators.

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